Homogeneity of ideals

Jacek Tryba

Hejnice, 30.01.2017

Joint work with Adam Kwela.

Homogeneity

The *restriction of the ideal* $\mathcal{I}$ *to* $X \subseteq \bigcup \mathcal{I}$ is given by

$$\mathcal{I}|X = \{A \cap X : \ A \in \mathcal{I}\}.$$

Given two ideals $\mathcal{I}$ and $\mathcal{J}$ we write $\mathcal{I} \cong \mathcal{J}$ if there is a bijection $f : \bigcup \mathcal{I} \longrightarrow \bigcup \mathcal{J}$ such that $f[C] \in \mathcal{J} \iff C \in \mathcal{I}$.

Homogeneity

The *restriction of the ideal* $\mathcal{I}$ *to* $X \subseteq \bigcup \mathcal{I}$ is given by

$$\mathcal{I}|X = \{A \cap X : \ A \in \mathcal{I}\}.$$

Given two ideals $\mathcal{I}$ and $\mathcal{J}$ we write $\mathcal{I} \cong \mathcal{J}$ if there is a bijection $f : \bigcup \mathcal{I} \longrightarrow \bigcup \mathcal{J}$ such that $f[C] \in \mathcal{J} \iff C \in \mathcal{I}$.

Definition (Homogeneity)

Let $\mathcal{I}$ be an ideal on ω. Then

$$H(\mathcal{I}) = \{A \subseteq \omega : \ \mathcal{I}|A \cong \mathcal{I}\}$$

is called *the homogeneity family of the ideal* $\mathcal{I}$.

Homogeneity

The *restriction of the ideal* $\mathcal{I}$ *to* $X \subseteq \bigcup \mathcal{I}$ is given by

$$\mathcal{I}|X = \{A \cap X : \ A \in \mathcal{I}\}.$$

Given two ideals $\mathcal{I}$ and $\mathcal{J}$ we write $\mathcal{I} \cong \mathcal{J}$ if there is a bijection $f : \bigcup \mathcal{I} \longrightarrow \bigcup \mathcal{J}$ such that $f[C] \in \mathcal{J} \iff C \in \mathcal{I}$.

Definition (Homogeneity)

Let $\mathcal{I}$ be an ideal on ω. Then

$$H(\mathcal{I}) = \{A \subseteq \omega : \ \mathcal{I}|A \cong \mathcal{I}\}$$

is called *the homogeneity family of the ideal* $\mathcal{I}$.

Theorem

When $A \in H(\mathcal{I})$ *and* $B \supseteq A$ *then* $B \in H(\mathcal{I})$.

Homogenous ideals

Definition

We call an ideal $\mathcal{I}$ on ω *homogeneous* if
$H(\mathcal{I}) = \mathcal{I}^+ = \{A \subseteq X : A \notin \mathcal{I}\}$.

Definition

We call an ideal $\mathcal{I}$ on ω *anti-homogeneous* if
$H(\mathcal{I}) = \mathcal{I}^\star = \{A \subseteq X : A^c \in \mathcal{I}\}$.

Examples

- Fin, the ideal of all finite sets, is homogeneous.

Examples

- Fin, the ideal of all finite sets, is homogeneous.
- Maximal ideals are both homogeneous and anti-homogeneous.

Examples

- Fin, the ideal of all finite sets, is homogeneous.
- Maximal ideals are both homogeneous and anti-homogeneous.
- Van der Waerden ideal $\mathcal{W} = \{A \subseteq \omega : \ \exists_{n \in \omega} A$ does not contain an arithmetic progression of length $n\}$ is homogeneous.

Examples

- Fin, the ideal of all finite sets, is homogeneous.
- Maximal ideals are both homogeneous and anti-homogeneous.
- Van der Waerden ideal $\mathcal{W} = \{A \subseteq \omega : \ \exists_{n\in\omega} A$ does not contain an arithmetic progression of length $n\}$ is homogeneous.
- Let $\{I_n : n \in \omega\}$ be a family of consecutive intervals such that each I_n has length $n!$. An ideal $\mathcal{I} = \{A \subseteq \omega : \lim_{n\to\infty} |A \cap I_n|/n! = 0\}$ is anti-homogeneous.

Examples

- Fin, the ideal of all finite sets, is homogeneous.
- Maximal ideals are both homogeneous and anti-homogeneous.
- Van der Waerden ideal $\mathcal{W} = \{A \subseteq \omega : \ \exists_{n \in \omega} A$ does not contain an arithmetic progression of length $n\}$ is homogeneous.
- Let $\{I_n : n \in \omega\}$ be a family of consecutive intervals such that each I_n has length $n!$. An ideal $\mathcal{I} = \{A \subseteq \omega : \lim_{n \to \infty} |A \cap I_n|/n! = 0\}$ is anti-homogeneous.
- Ideal of sets of asymptotic density zero $\mathcal{I}_d = \left\{A \subseteq \mathbb{N} : \lim_{n \to \infty} \frac{|A \cap \{0,1,\ldots,n\}|}{n+1} = 0\right\}$ is neither homogeneous nor anti-homogeneous.

Invariant functions

Definition (Balcerzak, Głąb, Swaczyna)

Let $\mathcal{I}$ be an ideal on ω and $f : \omega \to \omega$ be an injection. We say that f is:

- *$\mathcal{I}$-invariant* if $f[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$;
- *bi-$\mathcal{I}$-invariant* if $f[A] \in \mathcal{I} \iff A \in \mathcal{I}$ for all $A \subseteq \omega$.

If $f : \omega \to \omega$ is bi-$\mathcal{I}$-invariant then $f[\omega] \in H(\mathcal{I})$. On the other hand, if $A \in H(\mathcal{I})$ then there is a bi-$\mathcal{I}$-invariant $f : \omega \to \omega$ with $f[\omega] = A$.

Fix points of invariant functions

Theorem

The following are equivalent for any ideal $\mathcal{I}$ on ω:

- *there is an $\mathcal{I}$-invariant injection $f : \omega \to \omega$ with $\mathrm{Fix}(f) \notin \mathcal{I}^\star$ and $f[\omega] \notin \mathcal{I}$;*
- *$\mathcal{I}$ is not a maximal ideal.*

Theorem

The following are equivalent for any ideal $\mathcal{I}$ on ω:

- *there is a bi-$\mathcal{I}$-invariant injection $f : \omega \to \omega$ with $\mathrm{Fix}(f) \notin \mathcal{I}^\star$;*
- *there are $A, B \subseteq \omega$ such that $A \triangle B \notin \mathcal{I}$ and $\mathcal{I}|A \cong \mathcal{I}|B$.*

Fix points of invariant functions

Theorem

The following are equivalent for any ideal $\mathcal{I}$ on ω:

- *there is an $\mathcal{I}$-invariant injection $f : \omega \to \omega$ with $\mathrm{Fix}(f) \notin \mathcal{I}^\star$ and $f[\omega] \notin \mathcal{I}$;*
- *$\mathcal{I}$ is not a maximal ideal.*

Theorem

The following are equivalent for any ideal $\mathcal{I}$ on ω:

- *there is a bi-$\mathcal{I}$-invariant injection $f : \omega \to \omega$ with $\mathrm{Fix}(f) \notin \mathcal{I}^\star$;*
- *there are $A, B \subseteq \omega$ such that $A \triangle B \notin \mathcal{I}$ and $\mathcal{I}|A \cong \mathcal{I}|B$.*

Problem

Characterize the ideals for which there are no $A, B \subseteq \omega$ such that $A \triangle B \notin \mathcal{I}$ and $\mathcal{I}|A \cong \mathcal{I}|B$. Specifically, find a "nice" example of such an ideal.

Ideal convergence

Let $\mathcal{I}$ be an ideal on ω. We say that a real sequence $(x_n)_{n\in\omega}$ is *$\mathcal{I}$-convergent* to $x \in \mathbb{R}$ if for every $\varepsilon > 0$ we have

$$\{n \in \omega : |x_n - x| > \varepsilon\} \in \mathcal{I}.$$

Ideal convergence

Let $\mathcal{I}$ be an ideal on ω. We say that a real sequence $(x_n)_{n\in\omega}$ is *$\mathcal{I}$-convergent* to $x \in \mathbb{R}$ if for every $\varepsilon > 0$ we have

$$\{n \in \omega : |x_n - x| > \varepsilon\} \in \mathcal{I}.$$

Proposition

The following are equivalent for any ideal $\mathcal{I}$ on ω not isomorphic to $Fin \oplus \mathcal{P}(\omega)$:

- *for any sequence $(x_n)_{n\in\omega}$ of reals, $\mathcal{I}$-convergence of $(x_n)_{n\in\omega}$ to some $x \in \mathbb{R}$ implies convergence of $(x_{f(n)})_{n\in\omega}$ to x for some bi-$\mathcal{I}$-invariant injection f;*
- *for every countable family $\{A_n : \ n \in \omega\} \subseteq \mathcal{I}$ there exists such $A \in H(\mathcal{I})$ that $A \cap A_n$ is finite for every $n \in \omega$.*

A homogeneous ideal satisfies the above if and only if it is a weak P-ideal. Moreover, an anti-homogeneous ideal satisfies the above if and only if it is a P-ideal.

Enumeration as isomorphism

Theorem

Let $A \in H(\mathcal{I}_d)$ and $\{a_0, a_1 \dots\}$ be an increasing enumeration of A. Then the function $f : \omega \to A$ given by $f(n) = a_n$ witnesses that $\mathcal{I}_d | A \cong \mathcal{I}_d$.

Enumeration as isomorphism

Theorem

Let $A \in H(\mathcal{I}_d)$ and $\{a_0, a_1 \ldots\}$ be an increasing enumeration of A. Then the function $f : \omega \to A$ given by $f(n) = a_n$ witnesses that $\mathcal{I}_d | A \cong \mathcal{I}_d$.

Problem

Characterize ideals $\mathcal{I}$ such that for any $A \in H(\mathcal{I})$ the function $f : \omega \to A$ given by $f(n) = a_n$, where $\{a_0, a_1 \ldots\}$ is an increasing enumeration of A, witnesses that $\mathcal{I} | A \cong \mathcal{I}$.